On the cardinality of Hausdorff spaces

Filippo Cammaroto Andrei Catalioto Jack Porter

Dedicated to A.V. Arhangel'skiĭ *

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Abstract

A common generalization for two of the main streams of cardinality inequalities is developed; each stream derives from the famous inequality due to A.V. Arhangel'skiĭ in 1969 for Hausdorff spaces. Our result is used to obtain some of the increasing chain results of cardinality inequalities. The paper is concluded with some open problems.
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1 Introduction

Cardinal functions play a major role in classifying spaces. This is one of the reasons why research in cardinality inequalities has dramatically increased during the past half century. A fundamental boost in this research was a 1969 result by A.V. Arhangel'skiĭ ([2]) who proved that, for a Hausdorff space X, $|X| \leq 2^{L(X)\chi(X)}$. Since the appearance of this important result, there have been strings of improvements (see [12] for a detailed list) where each result in the string is an improvement of the previous result. At the vertex top of the multiple strings is Arhangel'skiĭ result. At the bottom of two separate strings are the following results for Hausdorff spaces X:

(1) Bella and Cammaroto (1988 - [6]): $|X| \leq 2^{aL_c(X)\psi_c(X)t(X)}$ and

(2) Bella (2012 - [4]): $|X| \leq 2^{L(X)F_c(X)\psi(X)}$.

The main theorem in this paper is a cardinality inequality that joins the ends of two strings by improving both (1) and (2). The development of the proof of this theorem starts with the Hodel's guidelines for understanding the standard proof (called the "closing-off argument") that are presented in the excellent paper by Hodel ([12]).

* On the occasion of his seventy-fifth birthday.

A strengthening of the cardinality inequality result is to show that if a Hausdorff space is the union of an increasing chain of spaces each satisfying the cardinality inequality with the same cardinal, then the Hausdorff space also satisfies the cardinality inequality. Many topologists, including Tkachenko, Juhász-Szentmiklóssy, Spadaro and Bella ([19, 14, 17, 18, 5]), have proved the increasing chains versions of most of the cardinal inequalities. One of the applications of our main result is a proof of these increasing chain versions. The paper is concluded with a few open problems.

2 Notations and terminologies

In this paper X will denote a topological space and $\tau(X)$ the topology on X. Our notation and terminology follow [10] for general topological notions, [3], [11], [13] for cardinal functions and [15] for H-closed spaces and H-closed extensions. We assume that the reader is familiar with the cardinal functions of L (Lindelöf), χ (character), ψ (pseudo-character), ψ_c (closed pseudo-character), and t (tightness). We start with a few definitions that may be new to the reader.

For $A \subseteq X$ and κ an infinite cardinal, let $[A]^{\leq\kappa} = \{B : B \subseteq A \text{ and } |B| \leq \kappa\}$ and for a collection $\mathcal{S}$ of subsets of X, let $\overline{\mathcal{S}} = \{cl_X S : S \in \mathcal{S}\}$. The *$\theta$-closure of A* is defined as $cl_\theta(A) = \{x \in X : cl_X U \cap A \neq \varnothing \text{ whenever, } x \in U \in \tau(X)\}$. A is *θ-closed* if $cl_\theta A = A$. If more then one space X is involved, $cl_\theta A$ is denoted by $cl_{\theta,X} A$ to prevent any possibility of confusion. The *κ-closure of A* is defined as $cl_\kappa(A) = \bigcup_{B\in[A]^{\leq\kappa}} cl_X B$. A set A is *κ-closed* if $cl_\kappa A = A$. It is immediate that $cl_\kappa(cl_\kappa A) = cl_\kappa A$. The *semiregularization* of a space X, denoted by X_s (or $X(s)$), is the set X with the topology generated by the family $RO(X) = \{U \in \tau(X) : U = int_X(cl_X(U))\}$ of regular open sets of X (if $X_s = X$, then X is called *semiregular*). Clearly, *every regular space is semiregular* (the converse is not true). For $x \in X$, the *θ-tightness of* x, denoted by $t_\theta(x, X)$, is the smallest infinite cardinal κ such that for each $A \subseteq X$ with $x \in cl_\theta A$, there is $B \in [A]^{\leq\kappa}$ such that $x \in cl_\theta B$. The *θ-tightness of* X, denoted by $t_\theta(X)$, is the supremum of the set $\{t_\theta(x, X) : x \in X\}$.

Hodel ([12]) divides the closing-off argument into three steps. The first two steps in his analysis of the closing-off argument are a combination of "tightness" and bounding the "closure" of a set. These two steps are satisfied when topological closure (cl_X) and χ are combined. In the setting of Urysohn spaces, these two steps are satisfied with the right combination of variations of cl_θ, ψ, and t_θ. A few topologists are trying to improve the cardinality inequality in Hausdorff spaces by seeking the right combination of variations of cl_θ and t_θ, that is, moving in the θ-direction. The concept of κ-closure, introduced in [9], combines both of Hodel's two steps into one step.

The tightness condition is built into the definition of κ-closure without actually requiring some variation of tightness. The third and final step of Hodel's analysis requires a "covering" cardinality. We will use a variation of "Lindelöf" as introduced in the next paragraph.

For $Y \subseteq X$, the *almost Lindelöf degree of* Y *relative to* X, denoted by $aL(Y,X)$, is the smallest infinite cardinal κ such that for every open cover $\mathcal{U}$ of Y by open sets in X, there is a subcollection $\mathcal{V} \in [\mathcal{U}]^{\leq\kappa}$ such that $Y \subseteq \bigcup \overline{\mathcal{V}}$. The *almost Lindelöf degree of* X, denoted by $aL(X)$, is $aL(X,X)$. The *almost Lindelöf degree relative to closed subsets of* X, denoted by $aL_c(X)$, is the supremum of the set $\{aL(C,X) : C \text{ closed subset of } X\}$. The κ-*almost Lindelöf degree of* X, denoted by $aL_\kappa(X)$, is the supremum of the set $\{aL(cl_\kappa A, X) : A \subseteq X\}$.

In proving (2), Bella ([4]) used the concept of $aL(A,X)$ when A is a κ-closed set A. This property, formally defined in the above paragraph, is used in our proof of the main result and corresponds to the third step in Hodel's analysis. Before some of the preliminaries properties are established, we will introduce variations of the concept of free sequences to better understand the relationship between (2) and Arhangel'skiĭ inequality.

For an infinite cardinal κ, a set $\{x_\alpha\}_{\alpha\in\kappa} \subseteq X$ is a *free sequence of length* κ if for every $\beta \in \kappa$,

$$cl_X\{x_\alpha : \alpha \leq \beta\} \cap cl_X\{x_\alpha : \alpha > \beta\} = \varnothing.$$

The *free sequence number of* X, denoted by $F(X)$, is the supremum of the infinite cardinals κ for which there exists a free sequence in X of length κ. Free sequences were used by Arhangel'skiĭ in his original proof.
A set $\{x_\alpha\}_{\alpha\in\kappa} \subseteq X$ is a θ-*free sequence of length* κ if for every $\beta \in \kappa$,

$$cl_\theta\{x_\alpha : \alpha \leq \beta\} \cap cl_\theta\{x_\alpha : \alpha > \beta\} = \varnothing.$$

The θ-*free sequence number of* X, denoted by $F_\theta(X)$, is the supremum of the infinite cardinals κ such that there exists a θ-free sequence in X of length κ.
A set $\{x_\alpha\}_{\alpha\in\kappa} \subseteq X$ is a c-*free sequence of length* κ if for each $\alpha \in \kappa$ there exists a collection $\mathcal{U}$ of open subsets of X such that $cl_X\{x_\beta : \beta < \alpha\} \subseteq \bigcup\mathcal{U}$ and $\bigcup\overline{\mathcal{U}} \cap \{x_\beta : \alpha \leq \beta < \kappa\} = \varnothing$. It is equivalent to say that $\{x_\alpha\}_{\alpha\in\kappa}$ is a c-free sequence of length κ if for every $\beta \in \kappa$,

$$cl_X\{x_\alpha : \alpha \leq \beta\} \cap cl_\theta\{x_\alpha : \alpha > \beta\} = \varnothing.$$

The c-*free sequence number of* X, denoted by $F_c(X)$, is the supremum of the infinite cardinals κ such that there exists a c-free sequence in X of length κ.
A subset $\{x_\alpha\}_{\alpha\in\kappa} \subseteq X$ is a a-*free sequence of length* κ if for every $\beta \in \kappa$,

$$cl_\theta\{x_\alpha : \alpha \leq \beta\} \cap cl_X\{x_\alpha : \alpha > \beta\} = \varnothing.$$

The *a-free sequence number of* X, denoted by $F_a(X)$, is the supremum of the infinite cardinals κ such that there exists a a-free sequence in X of length κ.

Let X be a Hausdorff space. Juhász and Spadaro (see [18]) proved that $|X| \leq 2^{L(X)F(X)\psi(X)}$ and this inequality improves Arhangel'skiĭ's inequality $|X| \leq 2^{L(X)\chi(X)}$ as $\psi(X) \leq \chi(X)$ and $F(X) \leq L(X)t(X)$. As $F_c(X) \leq F(X)$, Bella's inequality of $|X| \leq 2^{L(X)F_c(X)\psi(X)}$ improves the Juhász-Spardaro inequality. An interesting feature of Bella's inequality is the use of cl_θ in the equivalent formulation of F_c; Bella's result is move in the θ-direction. It is immediate that $F_\theta(X) \leq F_c(X) \leq F(X)$. The other half of the equivalent formulation is the definition of F_a; also $F_\theta(X) \leq F_a(X) \leq F(X)$.

In the comparison of $|X| \leq 2^{aL_c(X)\psi_c(X)t(X)}$ and $|X| \leq 2^{L(X)F_c(X)\psi(X)}$, it appears that there is a balancing relationship as $aL_c(X) \leq L(X)$ but $F_c(X) \leq aL_c(X)t(X)$. The main result of this paper targets a common intersection of these two cardinality inequalities.

3 Basic properties and examples

We assume the reader is familiar with the results in the following two propositions; the first five items in the first proposition are easy to verify and the last three items are known.

Proposition 1. Let X be a space, $Y \subseteq X$, and κ an infinite cardinal.

(a) $\psi(X) \leq \psi_c(X) \leq \chi(X)$, $t_\theta(X) \leq t(X) \leq \chi(X)$ and $aL(X) \leq aL_c(X) \leq L(X)$;

(b) $F_\theta(X) \leq F_c(X) \leq F(X)$ and $F_\theta(X) \leq F_a(X) \leq F(X)$;

(c) If X is regular, then $aL(X) = aL_c(X) = L(X)$, $t_\theta(X) = t(X)$ and $F_\theta(X) = F(X) = F_c(X) = F_a(X)$;

(d) $t(Y) \leq t(X)$;

(e) If Y is closed in X, then $F(Y) \leq F(X)$;

(f) ([13]) If X is Hausdorff, then $\psi_c(X) \leq L(X)\psi(X)$;

(g) ([3]) If X is Hausdorff, then $F(X) \leq L(X)t(X)$;

(h) ([4]) If X is Hausdorff, then $F_c(X) \leq aL_c(X)t(X)$.

A basic space that is useful as an example and sometimes as a counterexample is $\kappa\omega$, the *Katětov H-closed extension of* ω (which is H-closed and Urysohn). Also, letting $\beta\omega$ be the *Stone-Čech compactification of* ω, it is well-known that $(\kappa\omega)_s = \beta\omega$ (see *4.8* in [15] for details).

Example 1. Hodel shows that $aL(\beta\omega) = aL_c(\beta\omega) = L(\beta\omega) = \omega$ ([11]) and $aL(\kappa\omega) = \omega < aL_c(\kappa\omega) = \mathfrak{c} < L(\kappa\omega) = 2^{\mathfrak{c}}$ ([12]). Thus, for all κ, we have that $aL_\kappa(\beta\omega) = aL_\kappa(\kappa\omega) = \mathfrak{c}$. In [8] it is shown that $F_\theta(\kappa\omega) = \mathfrak{c} < F(\kappa\omega) = 2^{\mathfrak{c}}$. Also, in [4], Bella shows that $F_c(\kappa\omega) = \mathfrak{c}$. Now, we want to calculate $F_a(\kappa\omega)$. By *Proposition 1(b)*, we have that $F_a(\kappa\omega) \geq \mathfrak{c}$. Assume there is a a-free sequence $A = \{x_\alpha\}_{\alpha\in\mathfrak{c}^+}$ in $\kappa\omega$. As ω is countable, we can also assume that $A \subseteq \kappa\omega\backslash\omega$. For $\alpha < \mathfrak{c}^+$, let $A_\alpha = \{x_\beta : \beta \leq \alpha\}$ and $B_\alpha = \{x_\beta : \beta > \alpha\}$. Thus $cl_\theta A_\alpha \cap cl_{\kappa\omega} B_\alpha = \varnothing$. Now, $B_\alpha \subseteq \kappa\omega\backslash\omega$ is discrete and closed; so, $cl_{\kappa\omega} B_\alpha = B_\alpha$. As $\kappa\omega$ is H-closed, $cl_\theta A_\alpha$ is an H-set in $\kappa\omega$ and hence an H-set in $(\kappa\omega)_s = \beta\omega$. But H-sets in $\beta\omega$ are closed and compact. Thus, $\{cl_\theta A_\alpha : \alpha \in \mathfrak{c}^+\}$ is an increasing chain of closed subsets of $\beta\omega$. So, for each $\alpha \leq \mathfrak{c}^+$, $x_{\alpha+1} \in \beta\omega\backslash cl_\theta A_\alpha$ (which is open in $\beta\omega$). Since $\beta\omega$ has an open base $\mathcal{B}$ such that $|\mathcal{B}| = \mathfrak{c}$, for each $\alpha \in \mathfrak{c}^+$, there exists $U_\alpha \in \mathcal{B}$ such that $x_{\alpha+1} \in U_\alpha \subseteq \beta\omega\backslash cl_\theta A_\alpha$. If $\alpha < \beta < \mathfrak{c}^+$, $\alpha < \alpha + 1 \leq \beta$ and $x_{\alpha+1} \in cl_\theta A_{\alpha+1} \subseteq cl_\theta A_\beta$. So, $x_{\alpha+1} \notin \beta\omega\backslash cl_\theta A_\beta$ and hence $x_{\alpha+1} \notin U_\beta$. Thus, $U_\alpha \neq U_\beta$ and $\mathcal{B}$ has $\mathfrak{c}^+$ distincts elements. But this is not possible as $\mathcal{B}$ has size $\mathfrak{c}$. This completes the proof that $F_a(\kappa\omega) = \mathfrak{c}$. Combining all of the results, we have that $F_\theta(\kappa\omega) = F_c(\kappa\omega) = F_a(\kappa\omega) = \mathfrak{c} < F(\kappa\omega) = 2^{\mathfrak{c}}$.

Our next proposition provides the necessary results needed for the closing-off argument to work in the proof of the main theorem. It is the workhorse fact of this paper.

Proposition 2. Let X be a Hausdorff space, $A \subseteq X$, and κ an infinite cardinal.

(a) $cl_\kappa(cl_\kappa(A)) = cl_\kappa(A) \subseteq cl_X(A)$, and $aL_c(X) \leq aL_\kappa(X)$;

(b) If $\psi_c(X) \leq \kappa$ and $Y \subseteq X$, then $|cl_\kappa(Y)| \leq |Y|^\kappa$.

(c) If $\psi_c(X) \leq \kappa$, and $C \in [X]^{\leq\kappa}$, then $|cl_\kappa(C)| \leq 2^\kappa$;

(d) If $t(X) \leq \kappa$, then $cl_\kappa(A) = cl_X(A)$, and $aL_\kappa(X) = aL_c(X)$;

(e) If $aL_c(X)t(X) \leq \kappa$, then $aL_\kappa(X) \leq \kappa$;

(f) If $L(X)F(X) \leq \kappa$, and $A \subseteq X$, then $L(cl_\kappa(A)) \leq \kappa$;

(g) ([4]) If $L(X)F_c(X) \leq \kappa$, then $aL_\kappa(X) \leq \kappa$;

(h) $\chi(X)aL_c(X) \in \{\kappa : \kappa \geq \psi_c(X), \kappa \geq aL_\kappa(X)\}$.

Proof. (a) - Easy to verify.
(b) - For every $x \in X$, let $\{V(\alpha, x) : \alpha < \kappa\}$ be a collection of open neighbourhoods of x such that $\bigcap_{\alpha<\kappa} \overline{V(\alpha, x)} = \{x\}$. We have that $cl_\kappa(Y) = \bigcup_{A\in[Y]^{\leq\kappa}} cl_X(A)$ and then, for $x \in cl_\kappa Y$, there exists $A_x \in [Y]^{\leq\kappa}$ such that $x \in cl(A_x)$. Now, consider the family $\{A_x \cap V(\alpha, x) : \alpha < \kappa\}$; we have

$|A_x \cap V(\alpha,x)| \leq \kappa$ and $cl(A_x \cap V(\alpha,x)) = cl(cl(A_x) \cap V(\alpha,x))$. Consider the function $\Phi : cl_\kappa Y \to [[Y]^{\leq\kappa}]^{\leq\kappa}$ defined by $\Phi(x) = \{A_x \cap V(\alpha,x) : \alpha < \kappa\}$. Note that $\bigcap_{\alpha<\kappa} \overline{\Phi(x)} = \bigcap_{\alpha<\kappa}\{\overline{A_x \cap V(\alpha,x)} : \alpha < \kappa\} = \{x\}$ for any $x \in cl_\kappa Y$, and Φ is one-to-one. Thus $|cl_\kappa Y| \leq |[[Y]^{\leq\kappa}]^{\leq\kappa}| \leq |Y|^\kappa$.
(c) - Follows from (b).
(d) - Suppose $t(X) \leq \kappa$; the inclusion $cl_\kappa(A) \subseteq cl_X(A)$ is always true by definition. To show $cl_X(A) \subseteq cl_\kappa(A)$, let $x \in cl_X(A)$ and, as $t(X) \leq \kappa$, there exists $B \subseteq A$ such that $|B| \leq \kappa$ and $x \in cl_X(B)$. As $cl_\kappa(A) = \bigcup_{D\in[A]^{\leq\kappa}} cl_X(D)$, $x \in cl_\kappa(A)$. Also, $aL_\kappa(X) = aL_c(X)$.
(e) - Follows from (d).
(f) - By way of contradiction, let $\mathcal{U}$ be an open cover of A with no κ-subcover. We will define recursively $\{x_\alpha\}_{\alpha\in\kappa^+}$ and $\{\mathcal{V}_\alpha\}_{\alpha\in\kappa^+}$ so that $\{x_\alpha\}_{\alpha\in\kappa^+} \subseteq A$, $\mathcal{V}_\alpha \subseteq \mathcal{U}$, $cl\{x_\delta : \delta < \alpha\} \subseteq \bigcup \mathcal{V}_\alpha$, $|\mathcal{V}_\alpha| \leq \kappa$ for $\alpha < \kappa^+$ and $\mathcal{V}_\alpha \subseteq \mathcal{V}_\beta$ for $\alpha < \beta < \kappa^+$. Suppose $\beta < \kappa^+$ and suppose $\{x_\alpha\}_{\alpha<\beta}$ and $\{\mathcal{V}_\alpha\}_{\alpha<\beta}$ are defined. As $|\{x_\alpha\}_{\alpha<\beta}| \leq \kappa$, $cl(\{x_\alpha\}_{\alpha<\beta}) \subseteq cl_\kappa A = A$ and $cl(\{x_\alpha\}_{\alpha<\beta}) \subseteq \bigcup \mathcal{U}$. There exists $\mathcal{V}_\beta \subseteq \mathcal{U}$ such that $cl(\{x_\alpha\}_{\alpha<\beta}) \subseteq \bigcup \mathcal{V}_\beta$ with $|\mathcal{V}_\beta| \leq \kappa$ and $\mathcal{V}_\alpha \subseteq \mathcal{V}_\beta$ for $\alpha < \beta < \kappa^+$. So, $A \backslash \bigcup \mathcal{V}_\beta \neq \varnothing$ and let $x_\beta \in A \backslash \bigcup \mathcal{V}_\beta$. This completes the induction and $\{x_\alpha\}_{\alpha<\kappa^+}$ is defined. For $\beta < \gamma < \kappa^+$, we have that $x_\gamma \in A\backslash \bigcup \mathcal{V}_\gamma \subseteq A \backslash \bigcup \mathcal{V}_{\beta+1} \subseteq X \backslash \bigcup \mathcal{V}_{\beta+1}$. Thus, $cl(\{x_\gamma\}_{\gamma>\beta}) \subseteq X \backslash \bigcup \mathcal{V}_{\beta+1}$. As $cl(\{x_\gamma\}_{\gamma\leq\beta}) \subseteq \bigcup \mathcal{V}_{\beta+1}$, $cl(\{x_\gamma\}_{\gamma\leq\beta}) \cap cl(\{x_\gamma\}_{\gamma>\beta}) = \varnothing$. That is, $\{x_\alpha\}_{\alpha<\kappa^+}$ is a free sequence in X, a contradiction as $F(X) \leq \kappa$.
(g) - See [4].
(h) - If $\kappa = \chi(X)aL_c(X)$, then $\kappa \geq \psi_c(X)$ and $\kappa \geq t(X)$. By (d), it follows that $aL_\kappa(X) = aL_c(X) \leq \kappa$. $\square$

4 A cardinal inequality for Hausdorff spaces

By *Proposition 2(c)*, we note that the first two steps of Hodel's analysis are satisfied with the assumption of only $\psi_c(X) \leq \kappa$. The following main theorem shows that the third and final step of Hodel's analysis is satisfied using $aL_\kappa(X)$.

Theorem 1. *Let X be a Hausdorff space and κ an infinite cardinal such that $\psi_c(X) \leq \kappa$ and $aL_\kappa(X) \leq \kappa$. Then $|X| \leq 2^\kappa$.*

Proof. For $x \in X$, let $\{B(x,\alpha) : \alpha \in \kappa\} \subseteq \tau(X)$ such that $\bigcap_{\alpha\in\kappa} \overline{B(x,\alpha)} = \{x\}$. Let $H : \wp(X) \to X$ be a choice function where $H(\varnothing)$ is some arbitrary fixed point. For $A \subseteq X$ and $f : A \to \kappa$, let $\mathcal{G}(A,f) = \bigcup_{x\in A} \overline{B(x,f(x))}$. Also, note that $|\{\overline{\mathcal{G}(C,f)} : C \in [A]^{\leq\kappa}, f \in \kappa^C, \text{ and } A \subseteq \overline{\mathcal{G}(C,f)}\}| \leq |A|^\kappa \cdot \kappa^\kappa = |A|^\kappa \cdot 2^\kappa$. By transfinite induction, we define a sequence $\{A_\alpha : \alpha \in \kappa^+\}$ of κ-closed subsets of X. Let $A_0 = \{H(\varnothing)\}$ and, for $\beta \in \kappa^+$, suppose A_α is defined for each $\alpha < \beta$ such that $|A_\alpha| \leq 2^\kappa$ and, for $\alpha < \gamma < \beta$, $A_\alpha \subseteq A_\gamma$. Let $B = \{H(X \backslash \overline{\mathcal{G}(C,f)}) : C \in [\bigcup_{\alpha\in\beta} A_\alpha]^{\leq\kappa}, f \in \kappa^C, \text{ and } \bigcup_{\alpha\in\beta} A_\alpha \subseteq \overline{\mathcal{G}(C,f)}\}$. Then $|B| \leq |\bigcup_{\alpha\in\beta} A_\alpha|^\kappa \cdot 2^\kappa \leq (2^\kappa)^\kappa \cdot 2^\kappa = 2^\kappa$. Define $A_\beta = cl_\kappa(B \cup \bigcup_{\alpha\in\beta} A_\alpha)$.

Since $\psi_c(X) \leq \kappa$ and $|B \cup \bigcup_{\alpha\in\beta} A_\alpha| \leq 2^\kappa$, by *Proposition 2(b)*, $|A_\beta| \leq 2^\kappa$. As A_β is κ-closed, the set $A = \bigcup_{\alpha\in\beta} A_\alpha$ is κ-closed and $|A| \leq 2^\kappa$. We are done if we show that $X = A$. Assume, by way of contradiction, there is $p \in X\backslash A$. For each $a \in A$, there exists $f(a) \in \kappa$ such that $p \notin \overline{B(a, f(a))}$. Since $aL_\kappa(X) \leq \kappa$, $aL(A, X) \leq \kappa$ and there is $C \in [A]^{\leq\kappa}$ such that $A \subseteq \bigcup_{c\in C} \overline{B(c, f(c))} = \overline{\mathcal{G}(C, f)}$. So, there exists A_β such that $C \subseteq A_\beta$ and $H(X\backslash\overline{\mathcal{G}(C, f)}) \in A_{\beta+1} \subseteq A$. But, $H(X\backslash\overline{\mathcal{G}(C, f)}) \notin \overline{\mathcal{G}(C, f)} \supseteq A$, a contradiction. $\square$

By *Proposition 2(h)*, $\{\kappa : \kappa \geq \psi_c(X), \kappa \geq aL_\kappa(X)\} \neq \varnothing$. For a space X, the *almost Lindelöf pseudo-character of* X, denoted by $aL\psi(X)$, is defined to be $\min\{\kappa : \kappa \geq \psi_c(X) aL_\kappa(X)\}$.

Corollary 1. *For a Hausdorff space X, $|X| \leq 2^{aL\psi(X)}$.*

Note that in the proof of *Theorem 1*, we used only that $aL(cl_\kappa Y, X) \leq \kappa$ where $Y \subseteq X$ and $|Y| \leq 2^\kappa$. That is, if we define $aL^-_\kappa(X) = \sup\{aL(cl_\kappa Y, X) : Y \subseteq X, |Y| \leq 2^\kappa\}$, we can also show the following result with the same proof.

Corollary 2. *Let X be a Hausdorff space and κ an infinite cardinal such that $\psi_c(X) \leq \kappa$ and $aL^-_\kappa(X) \leq \kappa$. Then $|X| \leq 2^\kappa$.*

Corollary 3. *(Bella-Cammaroto, 1988 - [6])*
For a Hausdorff space X, $|X| \leq 2^{aL_c(X)\psi_c(X)t(X)}$.

Proof. Let $\kappa = aL_c(X)\psi_c(X)t(X)$.
We have that $\psi_c(X) \leq \kappa$ and, as $t(X) \leq \kappa$, we also have, by *Proposition 2(d)*, that $aL_\kappa(X) = aL_c(X) \leq \kappa$. Thus, $|X| \leq 2^\kappa = 2^{aL_c(X)\psi_c(X)t(X)}$. $\square$

Corollary 4. *(Bella, 2012 - [4])*
For a Hausdorff space X, $|X| \leq 2^{L(X)F_c(X)\psi(X)}$.

Proof. Let $\kappa = L(X)F_c(X)\psi(X)$.
By, *Proposition 1(f)*, we have that $\psi_c(X) \leq L(X)\psi(X) \leq \kappa$.
By *Proposition 2(g)*, as $L(X)F_c(X) \leq \kappa$, we have that $aL_\kappa(X) \leq \kappa$.
Thus, $|X| \leq 2^\kappa = 2^{L(X)F_c(X)\psi(X)}$. $\square$

5 A cardinal inequality for chains of spaces

We start with this simple fact.

Lemma 1. *Suppose X is Hausdorff, $Y \subseteq X$ and $aL\psi(X) \leq \kappa$. Then, for $p \in Y$, there exists $\mathcal{V} \in [\tau(X)]^{\leq\kappa}$ such that $\bigcap \overline{\mathcal{V}}^X \cap X = \bigcap \mathcal{V} \cap X = \{p\}$.*

Proof. As $\psi(Y) \leq \psi_c(Y) \leq \kappa$, there exists $\mathcal{U} \in [\tau(X)]^{\leq\kappa}$ such that $\bigcap \mathcal{U} \cap Y = \{p\}$. For each $y \in X \backslash \{p\}$, there is $U_y \in \tau(X)$ such that $y \in U_y$ and $p \notin cl_X U_y$. For $U \in \mathcal{U}$, we have that $Y \backslash U$ is closed (hence κ-closed) in Y. As $aL_\kappa(Y) \leq \kappa$ and $\{U_y \cap Y : y \in Y \backslash U\}$ is an open cover of $Y \backslash U$ by open sets in Y, there exists $A_U \in [Y \backslash U]^{\leq\kappa}$ such that $Y \backslash U \subseteq \bigcup_{y \in A_U} cl_Y(U_y \cap Y) \subseteq \bigcup_{y \in A_U} cl_X U_y$. So, let $\mathcal{V} = \{X \backslash cl_X U_y : y \in A_U, U \in \mathcal{U}\}$ and we have that $|\mathcal{V}| \leq \kappa$ and $\bigcap \overline{\mathcal{V}}^X \cap X = \bigcap \mathcal{V} \cap X = \{p\}$. ∎

Let X be a space with $X = \bigcup_{\alpha<\lambda} X_\alpha$ where $X_\beta \subseteq X_\alpha$ whenever $\beta < \alpha$. We write $X = \bigcup\!\!\uparrow_{\alpha<\lambda} X_\alpha$ to denote the *union of an increasing chain* of the spaces $\{X_\alpha : \alpha < \lambda\}$.

The next goal is to show that if $X = \bigcup\!\!\uparrow_{\alpha<\lambda} X_\alpha$ is Hausdorff and κ is infinite cardinal such that $aL\psi(X_\alpha) \leq \kappa$ for $\alpha < \lambda$, then $|X| \leq 2^\kappa$. This result will generalize some results by Tkachenko ([19]), Juhász-Szentmiklóssy ([14]), Spadaro ([17, 18]) and Bella ([5]). By *Theorem 1*, $|X_\alpha| \leq 2^\kappa$ for each $\alpha < \lambda$. In particular, if $\lambda \leq 2^\kappa$, we have that $|X| \leq 2^\kappa$. So, the proof of this result reduces to showing the statement when λ is regular and $2^\kappa < \lambda$. By replacing *Lemma 6.11* in [13] with our *Lemma 1*, we can use the same neat argument as Juhász to obtain that if $A \in [X]^{\leq\kappa}$, then $|cl_X A| \leq 2^\kappa$. In particular, if $Y \subseteq X$, then $cl_\kappa Y = \bigcup_{A \in [Y]^{\leq\kappa}} cl_X A$ and $|cl_\kappa Y| \leq |Y|^\kappa \cdot 2^\kappa = |Y|^\kappa$. Let $X = \bigcup\!\!\uparrow_{\alpha<\lambda} X_\alpha$. Call a set $Y \subseteq X$ *bounded* if $Y \subseteq X_\alpha$ for some α. It is clear that Y is bounded if and only if $|Y| \leq 2^\kappa$. Consequently, if $Y \subseteq X$ and $|Y| \leq 2^\kappa$, then $|cl_\kappa Y| \leq |Y|^\kappa 2^\kappa = 2^\kappa$, i.e. the κ-closure of a bounded set is bounded.

Lemma 2. *Let $X = \bigcup\!\!\uparrow_{\alpha<\lambda} X_\alpha$ be a Hausdorff space and κ be an infinite cardinal such that $aL\psi(X_\alpha) \leq \kappa$ for $\alpha < \lambda$ and $2^\kappa < \lambda$ with λ regular. Then $\psi_c(X) \leq \kappa$.*

Proof. Assume, by way of contradiction, that there is $p \in X_0$ such that $\psi_c(p, X) \geq \kappa^+$. By *Lemma 1*, for each $\alpha \in \lambda$, there exists $\mathcal{V}_\alpha \in [\tau(X)]^{\leq\kappa}$ such that $\bigcap \overline{\mathcal{V}}_\alpha^X \cap X_\alpha = \bigcap \mathcal{V}_\alpha \cap X_\alpha = \{p\}$. We will inductively define $\alpha : \kappa^+ \to \lambda$ and $p : \kappa^+ \to X$ such that for $\beta < \kappa^+$, $\overline{\{p_\nu : \nu < \beta\}}^X \subseteq X_{\alpha_\beta}$ and $p_\beta \in \bigcap \overline{\mathcal{W}}_{\alpha_\beta}^X \backslash X_{\alpha_\beta}$ where $\mathcal{W}_{\alpha_\beta} = \bigcup_{\nu \in \beta} \mathcal{V}_{\alpha_\nu}$. Suppose α_ν and p_ν are defined for $\nu < \xi < \kappa^+$. We have that the set $\overline{\{p_\nu : \nu < \xi\}}^X$ is bounded and contained in some X_{α_ξ} . Let $\mathcal{W}_{\alpha_\xi} = \bigcup_{\nu \in \xi} \mathcal{V}_{\alpha_\nu}$. There is some $p_\xi \in \bigcap_{W \in \mathcal{W}_\xi} \overline{W}^X \backslash X_{\alpha_\xi}$. This completes the induction. The set $S = \{p_\nu : \nu \in \kappa^+\}$ is bounded and $cl_\kappa S = \bigcup_{\nu \in \kappa^+} cl_X \{p_\delta : \delta < \nu\} \subseteq X_\gamma$ for some $\gamma \in \lambda$. So, there exists an open set $T \in \mathcal{V}_\gamma$ such that $|S \backslash T| = \kappa^+$. Now, $cl_\kappa S \backslash T = \bigcup_{\nu \in \kappa^+} cl_X \{p_\delta : \delta < \nu\} \backslash T$ is κ-closed and, for each $\nu < \kappa^+$, $cl_X \{p_\delta : \delta < \nu\} \backslash T \subseteq X_{\alpha_\nu}$. Since $\bigcap \overline{\mathcal{V}}_{\alpha_\nu} = \bigcap \mathcal{V}_{\alpha_\nu} = \{p\}$, then $cl_X \{p_\delta : \delta < \nu\} \backslash T \subseteq \bigcup \{X \backslash \overline{V} : V \in \mathcal{V}_{\alpha_\nu}\}$ and it follows that $cl_\kappa S \backslash T = \bigcup_{\nu \in \kappa^+} cl_X \{p_\delta : \delta < \nu\} \backslash T \subseteq \bigcup \{X \backslash \overline{V} : V \in \mathcal{V}_{\alpha_\nu}, \nu \in \kappa^+\}$.

As $aL_\kappa(X) \leq \kappa$, there is $\mathcal{W} \in [\bigcup_{\nu\in\kappa^+} \mathcal{V}_{\alpha_\nu}]^{\leq\kappa}$ such that $cl_\kappa S \backslash T \subseteq \bigcup\{\overline{X\backslash\overline{V}} : V \in \mathcal{W}\}$. There exists $\delta \in \kappa^+$ such that $\mathcal{W} \subseteq \bigcup_{\nu<\delta} \mathcal{V}_{\alpha_\nu} \subseteq \mathcal{W}_{\alpha_\delta}$. Also, there is $\xi > \delta$ such that $p_\xi \notin \bigcup\{\overline{X\backslash\overline{V}} : V \in \mathcal{V}_\delta\}$ and $p_\xi \in S\backslash T$. But this is a contradiction as $cl_\kappa S\backslash T \subseteq \bigcup\{\overline{X\backslash\overline{V}} : V \in \mathcal{W}\} \subseteq \bigcup\{\overline{X\backslash\overline{V}} : V \in \mathcal{V}_\delta\}$. □

Now we are ready to show our second main result:

Theorem 2. *Let $X = \bigcup\!\!\uparrow_{\alpha<\lambda} X_\alpha$ be a Hausdorff space and κ be an infinite cardinal such that $aL\psi(X_\alpha) \leq \kappa$ for $\alpha < \lambda$. Then $|X| \leq 2^\kappa$.*

Proof. As noted before *Lemma 2*, the proof reduces to showing the conclusion of the theorem when λ is regular and $2^\kappa < \lambda$. Now, by *Lemma 2*, $\psi_c(X) \leq \kappa$. Also, if $Y \subseteq X$ and $|Y| \leq 2^\kappa$, then $|cl_\kappa Y| \leq 2^\kappa$ and, for some $\alpha < \lambda$, $cl_\kappa Y \subseteq X_\alpha$. But we have that $aL_\kappa(X_\alpha) \leq \kappa$, thus $aL(cl_\kappa Y, X) \leq \kappa$. That is, $aL^-(X) \leq \kappa$. By *Corollary 2*, we have that $|X| \leq 2^\kappa$. □

Besides the usual increasing chain results, we have a new result for the Bella-Cammaroto's inequality.

Corollary 5. *Let $X = \bigcup\!\!\uparrow_{\alpha<\lambda} X_\alpha$ be a Hausdorff space and κ be an infinite cardinal such that $aL_c(X_\alpha) \cdot \psi_c(X_\alpha) \cdot t(X_\alpha) \leq \kappa$ for $\alpha < \lambda$. Then $|X| \leq 2^\kappa$.*

Proof. We have that $\psi_c(X_\alpha) \leq \kappa$ and, as $t(X_\alpha) \leq \kappa$, we also have, by *Proposition 2(d)*, that $aL_\kappa(X_\alpha) = aL_c(X_\alpha) \leq \kappa$. Thus, $|X| \leq 2^\kappa$. □

6 Open problems

We conclude with some interesting questions.

Question 1. *(Bella, 2012 - [4])*
Does $|X| \leq 2^{aL_c(X)F_c(X)\psi_c(X)}$ hold for every Hausdorff space X?

Question 2. *Does $|X| \leq 2^{L(X)F_a(X)\psi(X)}$ or $|X| \leq 2^{aL_c(X)F_a(X)\psi_c(X)}$ hold for every Hausdorff space X?*

Recently, in [7], Bonanzinga introduced the *Hausdorff number* $H(X)$ of a space X as follows:

$$H(X) := \min\{\kappa : \text{for } A \in [X]^{\geq\kappa}, \text{ there is } \{U_a : a \in A\} \subseteq \tau(X) \text{ such that } a \in U_a \text{ and } \bigcap_{a\in A} U_a = \varnothing\}.$$

Also, the author generalized some well-known cardinal inequalities for Hausdorff spaces in terms of the Hausdorff number.

Now, using $H(X)$, we also have the following questions:

Question 3. *Let X be a T_1-space, κ an infinite cardinal with $aL\psi(X) \leq \kappa$.*

(a) If $H(X) \leq \omega$, is it true that $|X| \leq 2^\kappa$?

(b) Is it true that $|X| \leq (H(X))^\kappa$ or $|X| \leq 2^{H(X)\cdot\kappa}$?

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Filippo Cammaroto
Andrei Catalioto
Dipartimento di Matematica
Università di Messina
Viale F. Stagno d'Alcontres 31
98166 Messina, Italia
[camfil,acatalioto]@unime.it

Jack Porter

Department of Mathematics
University of Kansas
1460 Jayhawk Blvd. (Snow Hall)
Lawrence, KS 66045-7594, USA
porter@math.ku.edu